Comment on "Maximal planar networks with large clustering coefficient and power-law degree distribution"

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This Comment corrects the error which appeared in the calculation of the degree distribution of random Apollonian networks [T. Zhou, G. Yan, and B. H. Wang, Phys. Rev. E. **71**, 046141 (2005)]. As a result, the expression of $P(k)$, which gives the probability that a randomly selected node has exactly k edges, has the form $P(k) \propto 1/[k(k+1)(k+2)]$.

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In a recent paper [1], Zhou *et al.* discussed a class of networks, called random Apollonian networks (RAN), which display scale-free degree distribution, very large clustering coefficient, and very small average path length. The simultaneous small-world and scale-free properties can mimic well many real-life network systems.

However, the analysis errs in calculating the degree distribution function. The rate equation they used to derive the expression of $P(k)$ has the following form:

$$n(N+1,k+1) = n(N,k)\frac{k}{N_\Delta} + n(N,k+1)\left(1 - \frac{k+1}{N_\Delta}\right), \quad (1)$$

where $n(N,k)$ denotes the number of nodes with degree k when N nodes are present in the network, and N_Δ, the total number of triangles. This rate equation describes the time evolution of the number of nodes with degree $k+1$. The first term accounts for the process in which a site with k links is connected to the new site, leading to a gain in the number of sites with $k+1$ links. This happens with probability $\frac{k}{N_\Delta}$. The second term on the right-hand side of Eq. (1) accounts for a corresponding role (loss). Using the asymptotic limit $n(N,k)=NP(k)$ and $P(k+1)-P(k)=dP/dk$, they obtained the form of the degree distribution function: $P(k) \propto k^{-\gamma}$ with $\gamma=(3N-5)/N$. However, as we will state below, that $\frac{k}{N_\Delta}$ does not give the right probability of nodes with degree k gaining a new link. Thus, their result for $P(k)$ is incorrect.

The growth of a RAN is performed by randomly selecting a triangle in the network, and then a new node connects to the three vertices of the triangle (see Ref. [1] for details). This means that the correlation among the nodes is very strong, i.e., after a new link is added to a node, the remaining two links are constrained to link to its two neighbors. Thus, on the one hand, $\frac{k}{N_\Delta}$ would include the probability of the evolution of other nodes whose degree is not equal to k; on the other hand, the evolution of the neighbors (except for those neighbors with degree k) of a node (this node has degree k) would effect its evolution in the reverse, which could lead to the information missing for the evolution of $n(N,k)$. The same analysis is also suitable for the second term of Eq. (1). In the following, we give out the right form of the expression of $P(k)$.

Instead of $n(N,k)$, let us investigate the time evolution of $p(k,t_i,t)$, the probability that at time t a node i introduced at time t_i has a degree k. This "node-based method" can overcome the strong node-node correlation arising in the "group-based method." Indeed, the growth of the RAN can be understood by the evolution of an adjacent network (AN): each triangle in the RAN is represented by an edge; whenever two triangles share a edge, the corresponding two edges are connected in the AN. That a new node is added to the RAN (leading to two triangles increasing) corresponds to a randomly selected edge being replaced by a new triangle in the AN (this leads to two edges increasing) [see Fig. 1]. Note that each edge in the AN represents three nodes in the original RAN. When a new node enters the system, the

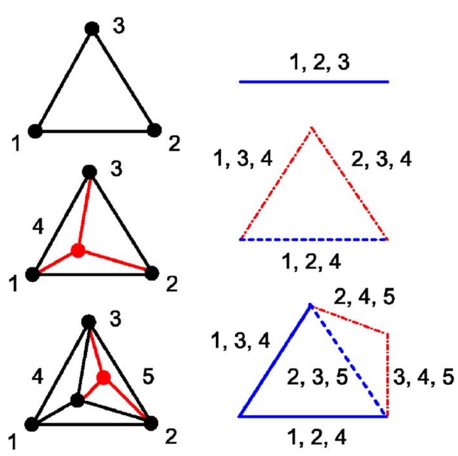

FIG. 1. (Color online) Schematic illustration of the growth of RAN (left column) and AN (right column) for $t=0$, 1 and 2 (from top to bottom). Each triangle in the RAN is represented by an edge in the AN. That a node added to the RAN corresponds to an edge is replaced by a triangle in the AN. Left: the (red, online) node and lines denote the newly added node and links. Right: the dashed (blue, online) line depicts the random selected edge (to update) and the dashed-dotted (red, online) lines denote the two newly added edges after each generation. Note that the information included in the edges are also renewed at the same time (the coordinates of nodes in the RAN denoted by the edges in the AN).

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degree of node i increases by 1 with a probability dependent on the emergence times of this node on all the edges of the AN, which is equal to the number of triangles containing the ith node in the RAN. Let N_Δ^i denote this number. For large time limit, N_Δ^i is equal to the degree of the ith node, $N_\Delta^i = k_i$.

Thus, for the ith node, the probability of gaining a new link (or the emergence times increasing by 1) is $k_i/2t$, where $2t$ denotes the total edges in the AN at time t (note that each updating step would give rise to two extra edges increasing). Consequently, the master equation governing $p(k,t_i,t)$ for the RAN has the form [2,3]

$$p(k,t_i,t+1) = \frac{k-1}{2t} p(k-1,t_i,t) + \left(1 - \frac{k}{2t}\right) p(k,t_i,t), \quad (2)$$

with the initial condition $p(k,t_i=0,1,t=1) = \delta_{k,1}$ and the boundary one $p(k,t,t) = \delta_{k,1}$. The degree distribution can be obtained as [2]

$$P(k) = \lim_{t\to\infty} \frac{[\sum_{t_i} p(k,t_i,t)]}{t}. \quad (3)$$

Using Eq. (2) and the expression $p(k) - p(k-1) = dp/dk$, one can get that $P(k)$ is the solution of the recursive equation: $P(k) = P(k-1)(k-1)/(k+2)$ for $k \geq m+1$, and $P(m) = 2/(m+2)$ for $k=m$, where m is the degree of a node at the time it enters the sytem (in the present case $m=3$). Solving for $P(k)$ gives that

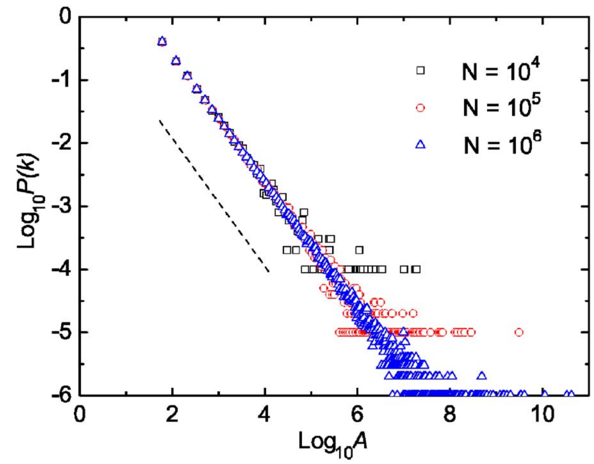

FIG. 2. (Color online) The distribution of $P(k)$ as a function of A, where $A = k(k+1)(k+2)$, for different network sizes. The dashed line is a guide to the eyes with slope -1.0.

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}. \quad (4)$$

In Fig. 2, the simulation results of $P(k)$ for different network sizes $N = 10^4$, 10^5, and 10^6 are plotted as a function of $k(k+1)(k+2)$; the best fit line gives out that the decay exponent is -1.002 ± 0.003, which is well in agreement with the analytic result Eq. (4).

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